9/3/25

Lecture 5

e.g. Matrix Multiplication

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\vec{x}} = 2\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

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e.g. Let $O_{m \times n}$ = $m \times n$ matrix whose entries are all 0

observe: If $A = m \times n$ matrix and $\vec{x} \in \mathbb{R}^n$ then:

$$A(\vec{0}_n) = \vec{0}_m \quad \text{and} \quad O_{m \times n}\vec{x} = \vec{0}_m$$

Let $I_{n \times n}$ = $n \times n$ matrix $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}$

$$I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

main diagonal

$$I_{n \times n}\vec{x} = x_1\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + x_n\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \vec{x}$$

* If you multiply some vector by the right sized identity matrix then you get the same vector out

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$\boxed{\text{Ex}}$ $\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}4\\5\\6\end{pmatrix} = \begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix}$

$$= \begin{bmatrix}1\\2\\3\end{bmatrix}^T \begin{bmatrix}4\\5\\6\end{bmatrix}$$

Rmk: If A is an $m \times n$ matrix the transpose of A is the $n \times m$ matrix A^T whose i^{th} row is the i^{th} column of A (and vice versa)

$\boxed{\text{Ex}}$ $\begin{bmatrix}1 & 4\\2 & 5\\3 & 6\end{bmatrix}^T = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$

Verifying $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$

Write $\vec{x} = \begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$ and $\vec{y} = \begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix}$

Then OTOH

$$A(\vec{x}+\vec{y}) = A\begin{pmatrix}x_1+y_1\\\vdots\\x_n+y_n\end{pmatrix} = (x_1+y_1)\vec{a}_1 + \dots + (x_n+y_n)\vec{a}_n$$

where $\vec{a}_1, \dots, \vec{a}_n \in \mathbb{R}^m$ are the columns of A

OTOH

$$A\vec{x} + A\vec{y} = (x_1\vec{a}_1 + \dots + x_n\vec{a}_n) + (y_1\vec{a}_1 + \dots + y_n\vec{a}_n)$$
$$= (x_1+y_1)\vec{a}_1 + \dots + (x_n+y_n)\vec{a}_n$$
$$= A(\vec{x}+\vec{y})$$

Match!

$\blacksquare$

$m \times n$ Linear system can be expressed $A\vec{x} = \vec{b}$ ← given Vector

Matrices + functions

$f(x) = e^x$

$f : \mathbb{R} : \mathbb{R}$ "# goes in, # comes out"

Def: Let A + B be sets

e.g. $A = \{$ND students on campus$\}$

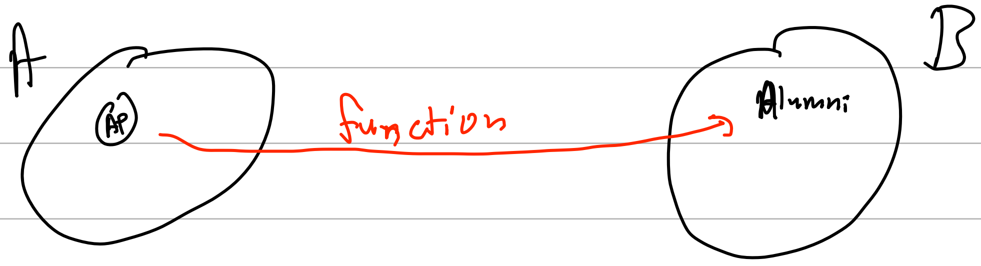
$B = \{$dorms on campus$\}$

function $f : A \to B$ is a "rule" to each $a \in A$ an element $f(a) \in B$

A is the domain (or source) of F

B is the codomain (or target)

Let $f(a)$ = dorm a lives in



An $m \times n$ matrix A gives a function

$$T_A : \mathbb{R}^n \to \mathbb{R}^m$$

defined by $T_A(\vec{x}) = A\vec{x}$

Proposition (restated)

(i) $T_A(\vec{x}+\vec{y}) = T_A(\vec{x}) + T_A(\vec{y})$

(ii) $cT_A(\vec{x}) = T_A(c\vec{x})$